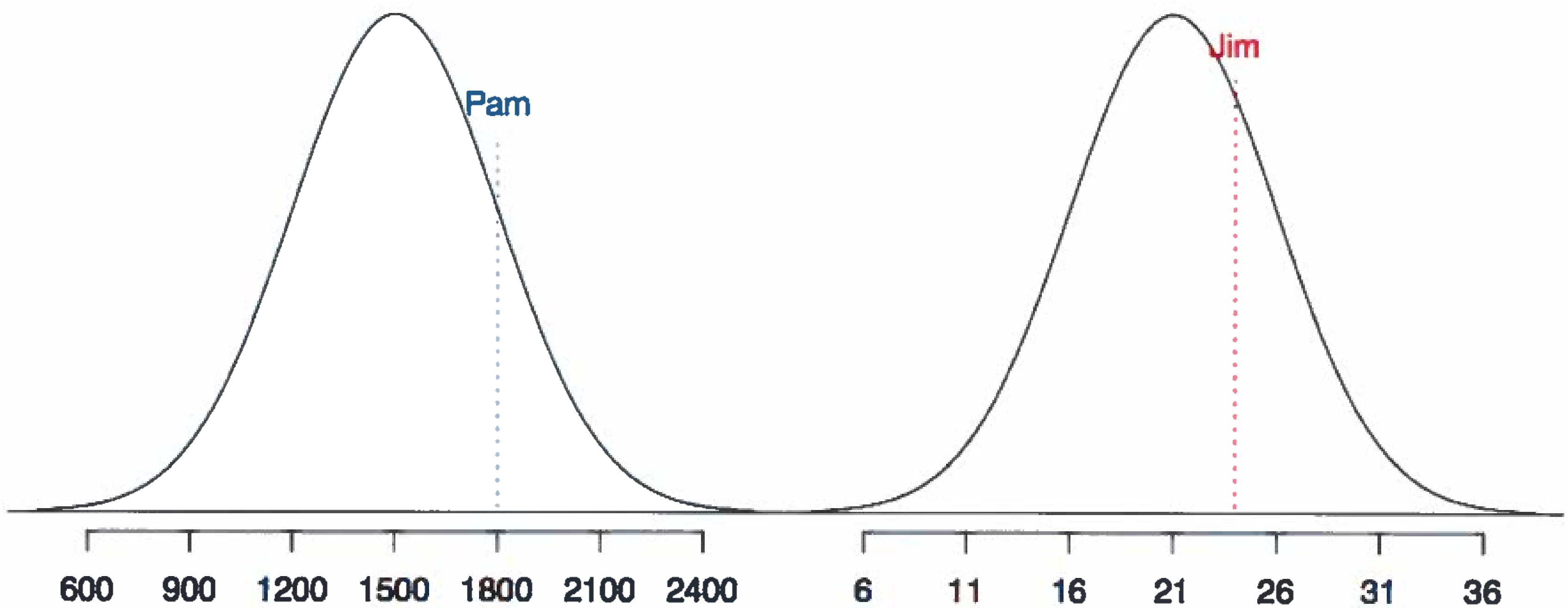
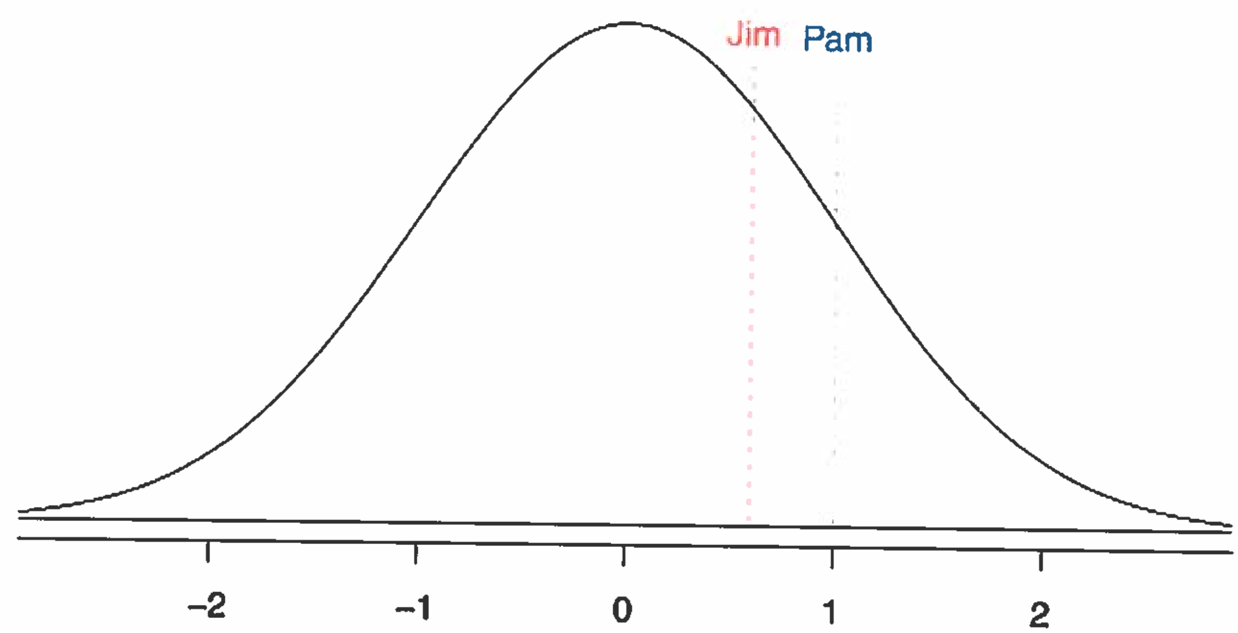
SAT scores are distributed nearly normally with mean 1500 and standard deviation 300. ACT scores are distributed nearly normally with mean 21 and standard deviation 5. A college admissions officer wants to determine which of the two applicants scored better on their standardized test with respect to the other test takers: Pam, who earned an 1800 on her SAT, or Jim, who scored a 24 on his ACT?



Standardizing with Z scores

Since we cannot just compare these two raw scores, we instead compare Z scores, how many standard deviations above or below the mean each observation is.

- Pam's score is (1800 - 1500) / 300 = 1 standard deviation above the mean.
- Jim's score is (24 - 21) / 5 = 0.6 standard deviations above the mean.



Standardizing with Z scores (cont.)

These are called standardized scores, or Z scores.

- Z score of an observation is the number of standard deviations it falls above or below the mean.

  $$Z = (\text{observation} - \text{mean}) / SD$$

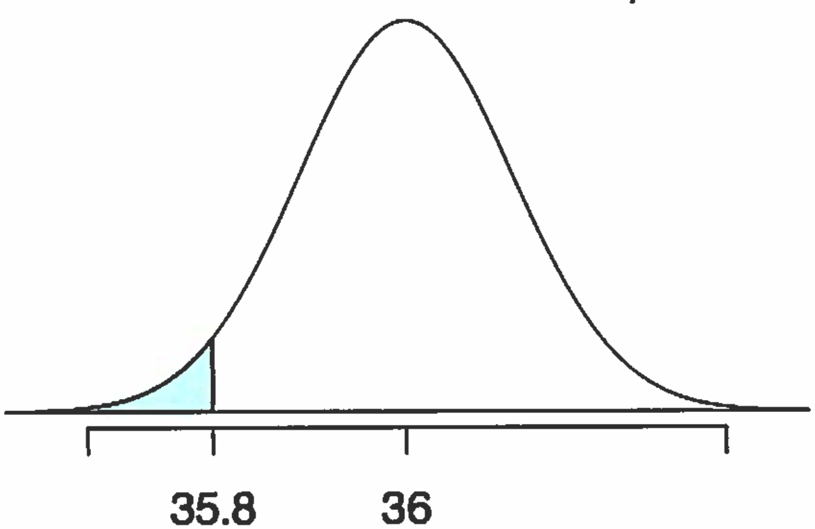
- We can use Z scores to roughly identify which observations are more unusual than others
- Z scores are defined for distributions of any shape,
- Z scores can be used to calculate percentiles for normal distributions only

Answer: .0608
Solution: Obtain the Z-scores for direct comparison
Pam $$\text{Z-score} = \frac{1800-1500}{300} = 1.00$$
Jim $$\text{Z-score} = \frac{24-21}{5} = 0.6$$

Example

At Heinz ketchup factory the amounts which go into bottles of ketchup are supposed to be normally distributed with mean 36 oz. and standard deviation 0.11 oz. Once every 30 minutes a bottle is selected from the production line, and its contents are noted precisely. If the amount of ketchup in the bottle is below 35.8 oz. or above 36.2 oz., then the bottle fails the quality control inspection. What percent of bottles have less than 35.8 ounces of ketchup?

- *Let X = amount of ketchup in a bottle: $X \sim N(\mu = 36, \sigma = 0.11)$*

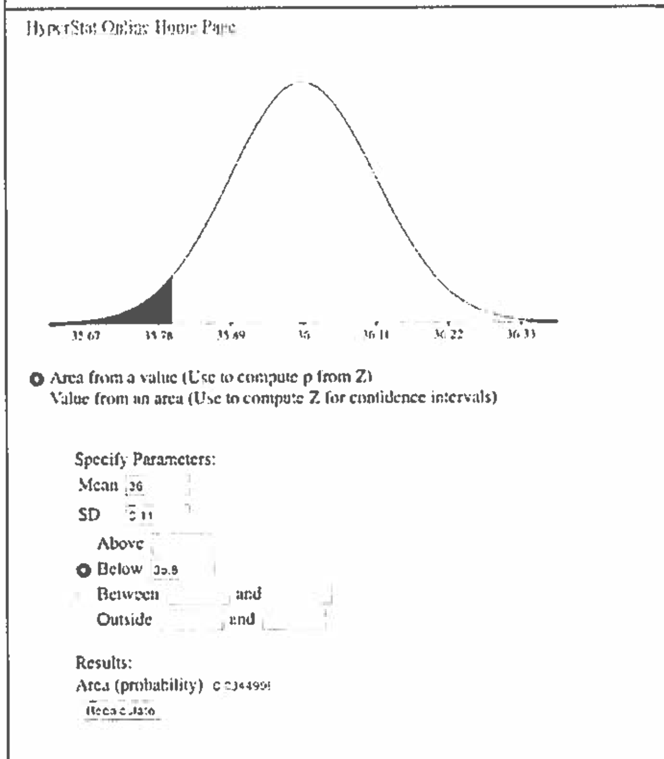


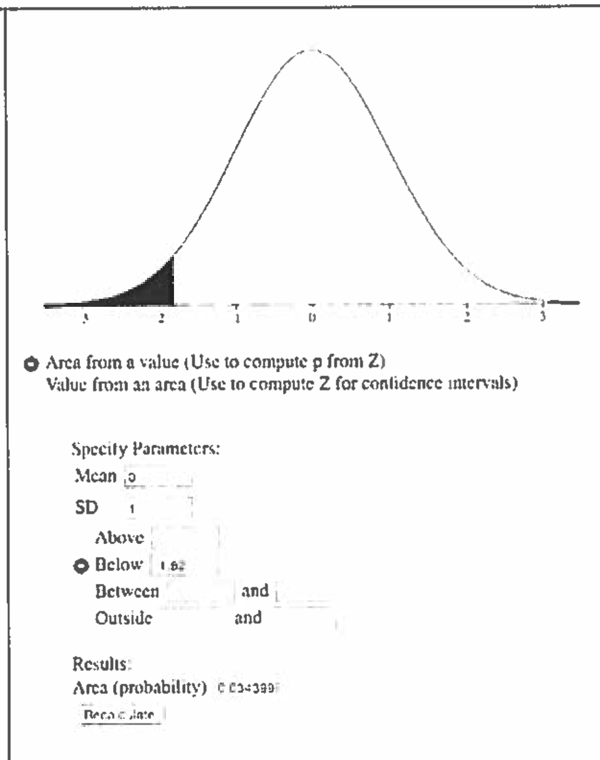
$$Z = \frac{35.8 - 36}{0.11} = -1.82$$

Answer: 3.4%

Solution:

$$\text{Z-score} = \frac{35.8-36}{0.11} = -1.82$$

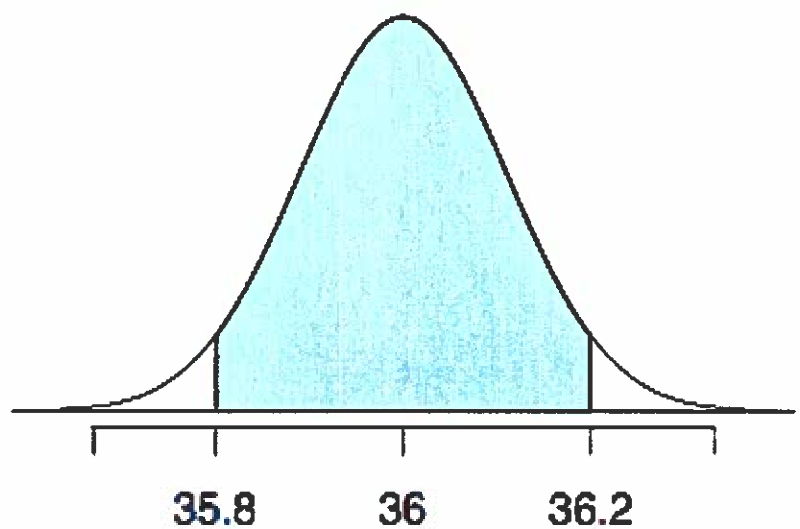




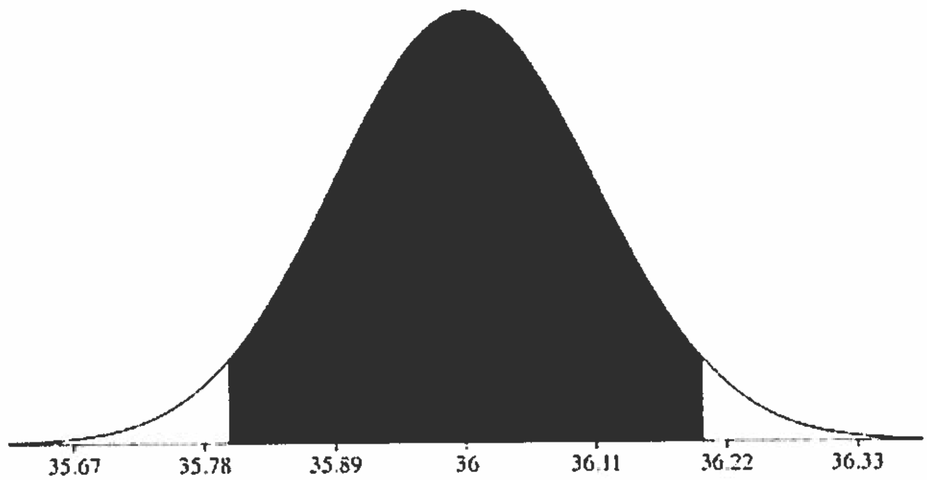
Finding probabilities within an interval

We that 96.6% of bottle (1-3.4) are more than 35.8 oz., but in order to pass inspection bottles need to also be less than 36.2 oz.

What percent of bottles pass the quality control inspection (i.e. 35.8 oz. < x < 36.2 oz.)?



Answer: 93.1%



- (●) Area from a value (Use to compute p from Z)
- ( ) Value from an area (Use to compute Z for confidence intervals)

Specify Parameters:

Mean 36

SD 0.11

( ) Above

( ) Below

(●) Between 35.8 and 36 2

( ) Outside and

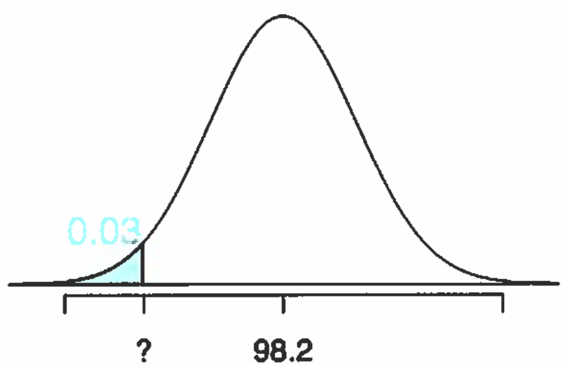
Results:

Area (probability) 0 931

[Recalculate]

Finding cutoff points

Body temperatures of healthy humans are distributed nearly normally with mean 98.2°F and standard deviation 0.73°F. What is the cutoff for the lowest 3% of human body temperatures?



0.09	0.08	0.07	0.06	0.05	Z
0.0233	0.0239	0.0244	0.0250	0.0256	−1.9
0.0294	0.0301	0.0307	0.0314	0.0322	−1.8
0.0367	0.0375	0.0384	0.0392	0.0401	−1.7

$$P(X < x) = 0.03 \rightarrow P(Z < -1.88) = 0.03$$

$$Z = \frac{obs - mean}{SD} \rightarrow \frac{x - 98.2}{0.73} = -1.88$$

$$x = Z \times SD + mean = (-1.88 \times 0.73) + 98.2 = 96.83^{o} \quad F$$

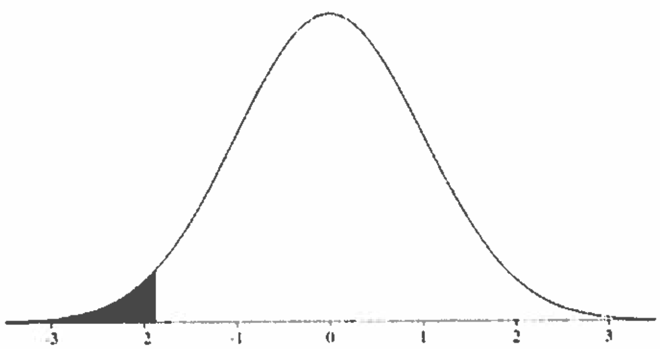
Mackowiak, Wasserman, and Levine (1992), A Critical Appraisal of 98.6 Degrees F, the Upper Limit of the Normal Body Temperature, and Other Legacies of Carl Reinhold August Wunderlick.

Answer: 96.83

Step 1:
Obtain z-score value of the 3rd percentile = -1.88

Step 2:
Solve for 3rd percentile of X = Human body temperature

$$\text{X-tile}_{.03} = \sigma * \text{Z-tile}_{.03} + \mu = (-1.88*0.73) + 98.2 = 96.83$$



Area from a value (Use to compute p from Z)
Value from an area (Use to compute Z for confidence intervals)

Specify Parameters:
Area 0.03
Mean 0
SD 1

Results:
Recalculate
Above
Below -1.881
Between
Outside

Practice

A census of persons recovering from lower-extremity fractures find they work an average of 8 hours per week, with a standard deviation of 2 hours per week, while in the first 6 months of recovery. One year post-injury, these persons work an average of 12 hours per week, with a standard deviation of 3.5 hours per week.

Practice

1. What proportion of persons work at least 10 hours per week during the first 6 months of recovery?
2. What proportion of persons work at least 10 hours per week one year post-recovery
3. What is the median number of hours worked during the first 6 months of recovery?
4. What is the 90^{th} percentile in the number of hours worked per week for persons one year post-injury?
5. Between how many hours per week does approximately 68% of the persons work one year post-injury?

Word Problem #1 (Normal Distribution)

Suppose that the distribution of diastolic blood pressure in a population of hypertensive women is modeled well by a normal probability distribution with mean 100 mm Hg and standard deviation 14 mm Hg. Let X be the random variable representing this distribution. Find two symmetric values "a" and "b" such that

$$\text{Probability} [a < X < b] = .99$$

Word Problem #2 (Normal Distribution)

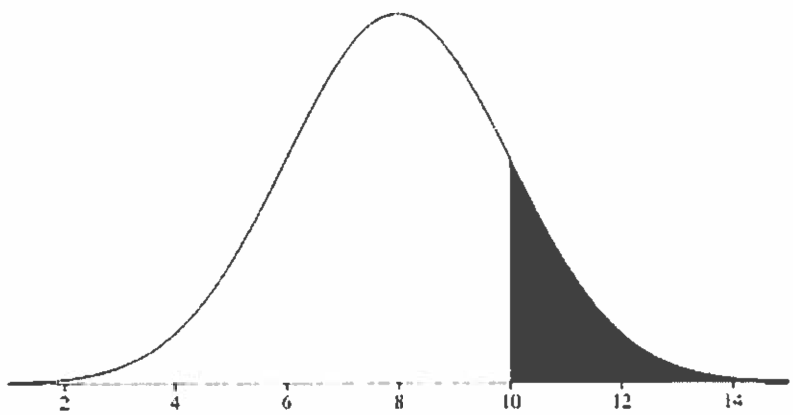
Suppose that the distribution of weights of New Zealand hamsters is distributed normal with mean 63.5 g and standard deviation 12.2 g. If there are 1000 weights in this population, how many of them are 78 g or greater?

Word Problem #3 (Normal Distribution)

Consider again the normal probability distribution of problem #2. What is the probability of selecting at random a sample of 10 hamsters that has a mean greater than 65 g?

#1

Answer: 15.9%



Area from a value (Use to compute p from Z)

Value from an area (Use to compute Z for confidence intervals)

Specify Parameters:

Mean 8

SD 2

Above 10

Below

Between and

Outside and

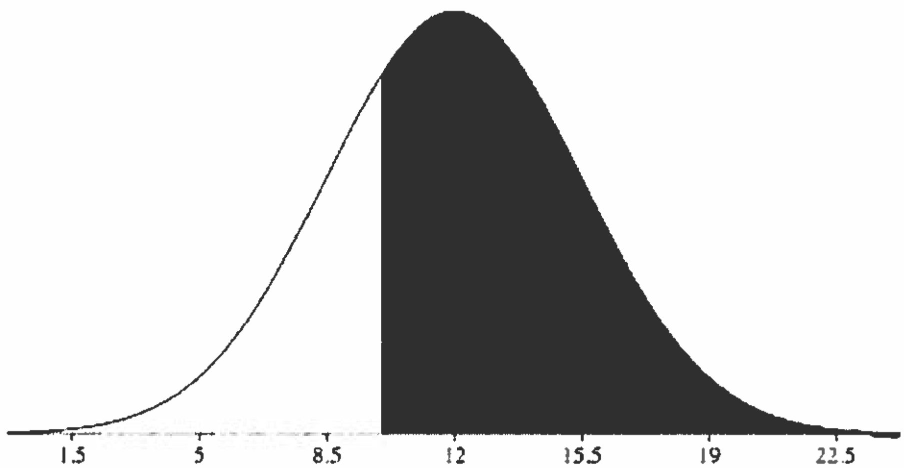
Results:

Area (probability) 0.1587

Recalculate

#2
Answer: 71.6%

HyperStat Online Home Page



- (•) Area from a value (Use to compute p from Z)
- ( ) Value from an area (Use to compute Z for confidence intervals)

Specify Parameters:

Mean [12]

SD [3.5]

(•) Above [10]

( ) Below [ ]

( ) Between [ ] and [ ]

( ) Outside [ ] and [ ]

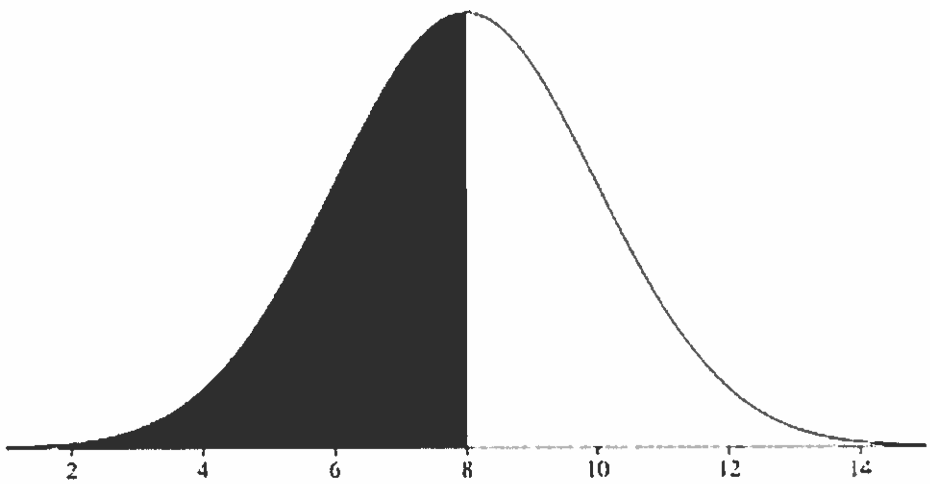
Results:

Area (probability) 0.7161

[Recalculate]

#3

Answer: 8 hours



( ) Area from a value (Use to compute p from Z)

(•) Value from an area (Use to compute Z for confidence intervals)

Specify Parameters:

Area	.50
Mean	8
SD	2

Results:

[Recalculate]

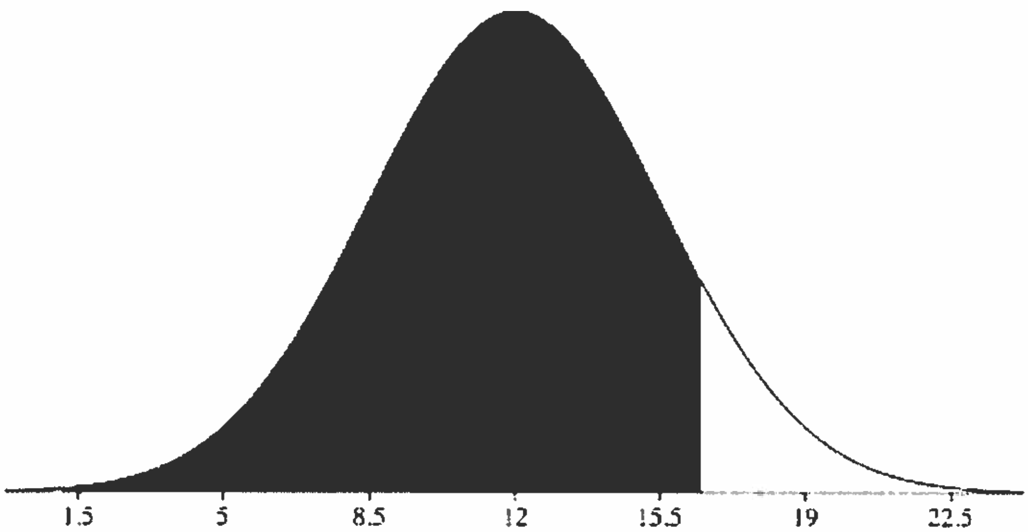
( ) Above

(•) Below 8

( ) Between

( ) Outside

#4
Answer: 16.86



- Area from a value (Use to compute p from Z)
- Value from an area (Use to compute Z for confidence intervals)

Specify Parameters:

Area .90

Mean 12

SD 3.5

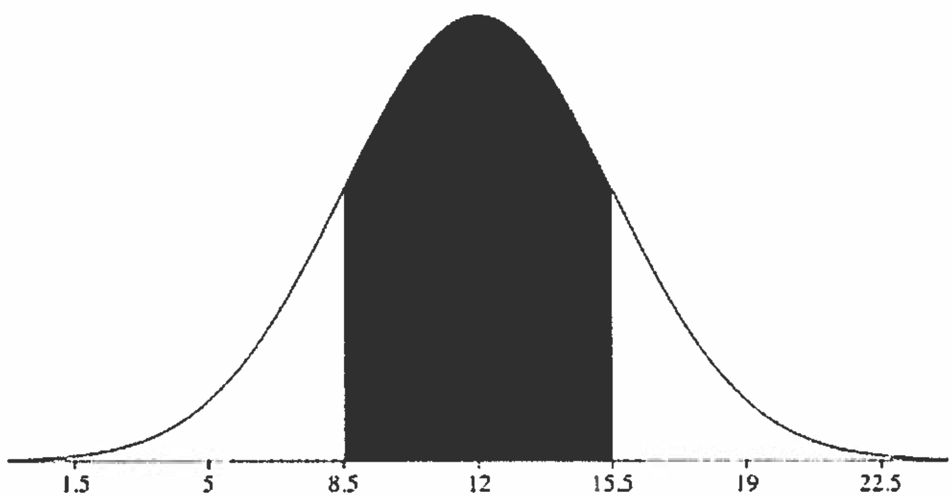
Results:

Recalculate

- Above
- Below 16.486
- Between
- Outside

#5

Answer: Between 8.52 and 14.48



Area from a value (Use to compute p from Z)

Value from an area (Use to compute Z for confidence intervals)

Specify Parameters:

Area	.68
Mean	12
SD	3.5

Results:

Recalculate

- Above
- Below
- Between 8.52 and 15.48
- Outside

Word Problem #1 (Normal Distribution) – SOLUTION

Answer: a=63.95 b=136.05

Easy (but not as thoughtful) Solution:

Step 1

Launch the David Lane normal distribution calculator provided to you on the topic page (5. Normal) of the course website: **http://davidmlane.com/hyperstat/Z_table.html**

Step 2

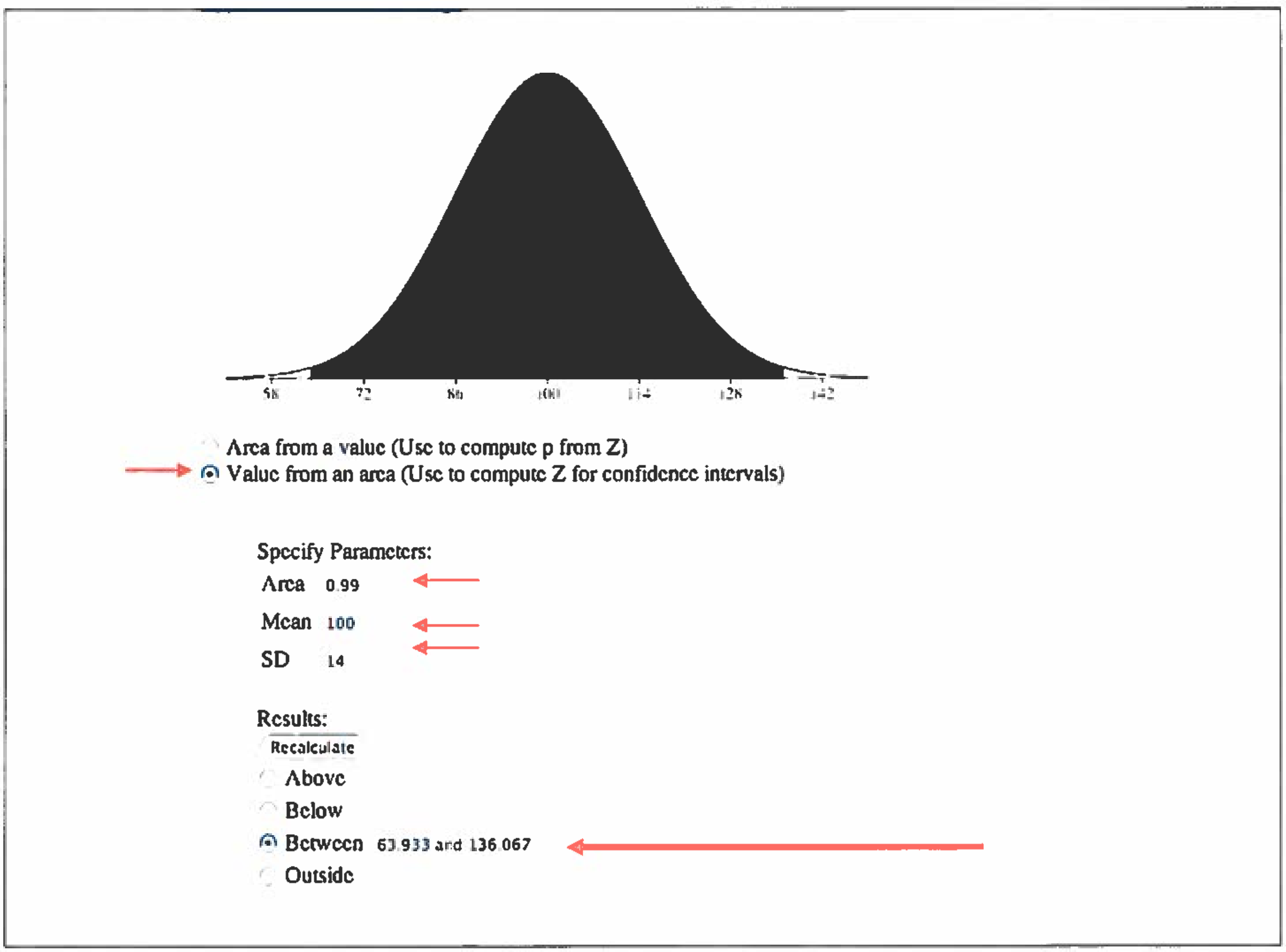
Click **on** the radio button to select, **"Value from an area (Use to compute Z for confidence intervals)"**

Step 3

In the box, labeled **area**, enter the value **.99,** in the box labeled **mean**, enter **100**, in the box labeled **SD** enter **14.**

Step 4

Click **on** the radio button to select, **"Between"**



Solution Using Z-Score:

Step 1

Launch the David Lane normal distribution calculator provided to you on the topic page (5. Normal) of the course website: **http://davidmlane.com/hyperstat/Z_table.html**

Step 2

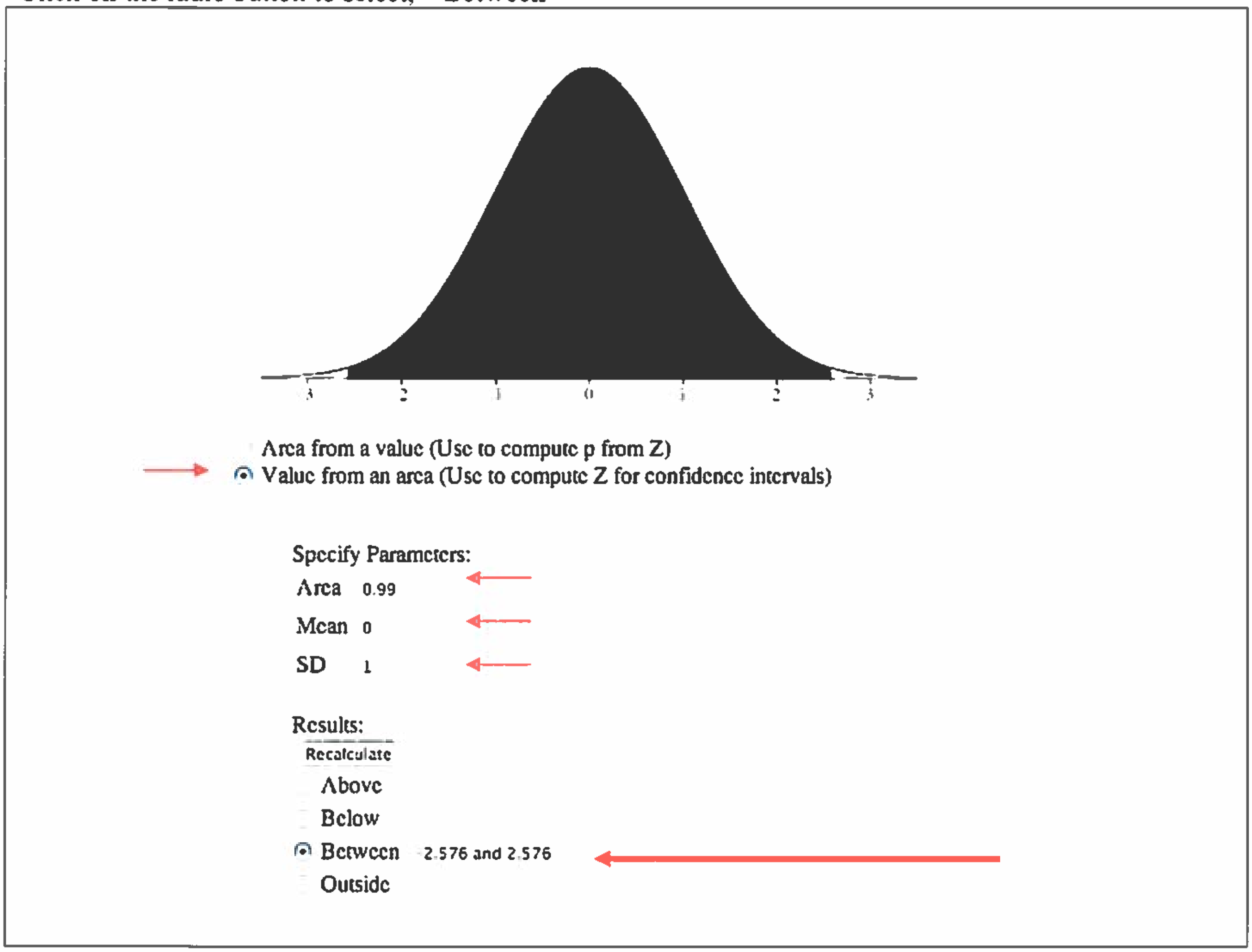
Click **on** the radio button to select, **"Value from an area (Use to compute Z for confidence intervals)"**

Step 3

In the box, labeled **area**, enter the value **.99,** in the box labeled **mean**, enter **0**, in the box labeled **SD** enter **1**.

Step 4

Click **on** the radio button to select, **"Between"**



Step 5

From the 0.5th and 99.5th percentiles of the standard normal distribution, solve for the corresponding values of the normal distribution that has mean=100 and sd=14.

Tip - Notice that the 0.5th and 99.5th percentiles are -2.57 and +2.57, symmetric about zero. So, really, we only needed to solve for one of them.

$$z = \frac{x-\mu}{\sigma} \text{ says that } \quad x=\sigma[z] + \mu$$

Thus a = 0.5th percentile for X = 14[-2.57] + 100 = 63.95
and b = 99.5th percentile for X = 14[+2.57] + 100 = 136.05

Word Problem #2 (Normal Distribution) - SOLUTION
Answer: 117

Solution:

Pr [weight > 78 g] = Pr [Normal μ=63.5 σ=12.2 > 78]

$$= \Pr\left[\text{ Standard normal } > \frac{78-\mu}{\sigma}\right] = \Pr\left[\text{ Standard normal } > \frac{78-63.5}{12.2}\right]$$

= Pr [Normal (0,1) > 1.1885]

= .117

Therefore # Hamsters with weights > 78 g in a population of size 1000

= (1000)(.117)
= 117

Word Problem #3 (Normal Distribution) - SOLUTION

Answer: .3483

Easy Solution:

The solution to this problem requires noticing that the random variable is $\overline{X}$, so that the standardization to Z must use the SE of $\overline{X} = \sigma / \sqrt{n}$. *Tip* - But the David Lane calculator does not have a box for you labeled SE. It has only the box labeled SD. This is okay, however. Simply provide the value of the SE in the SD box.

Step 1

Solve for the value of the standard error of the sample mean. SE = $\sigma / \sqrt{n}$ = $12.2 / \sqrt{10} = 12.2 / 3.16$ = **3.86**

Step 2

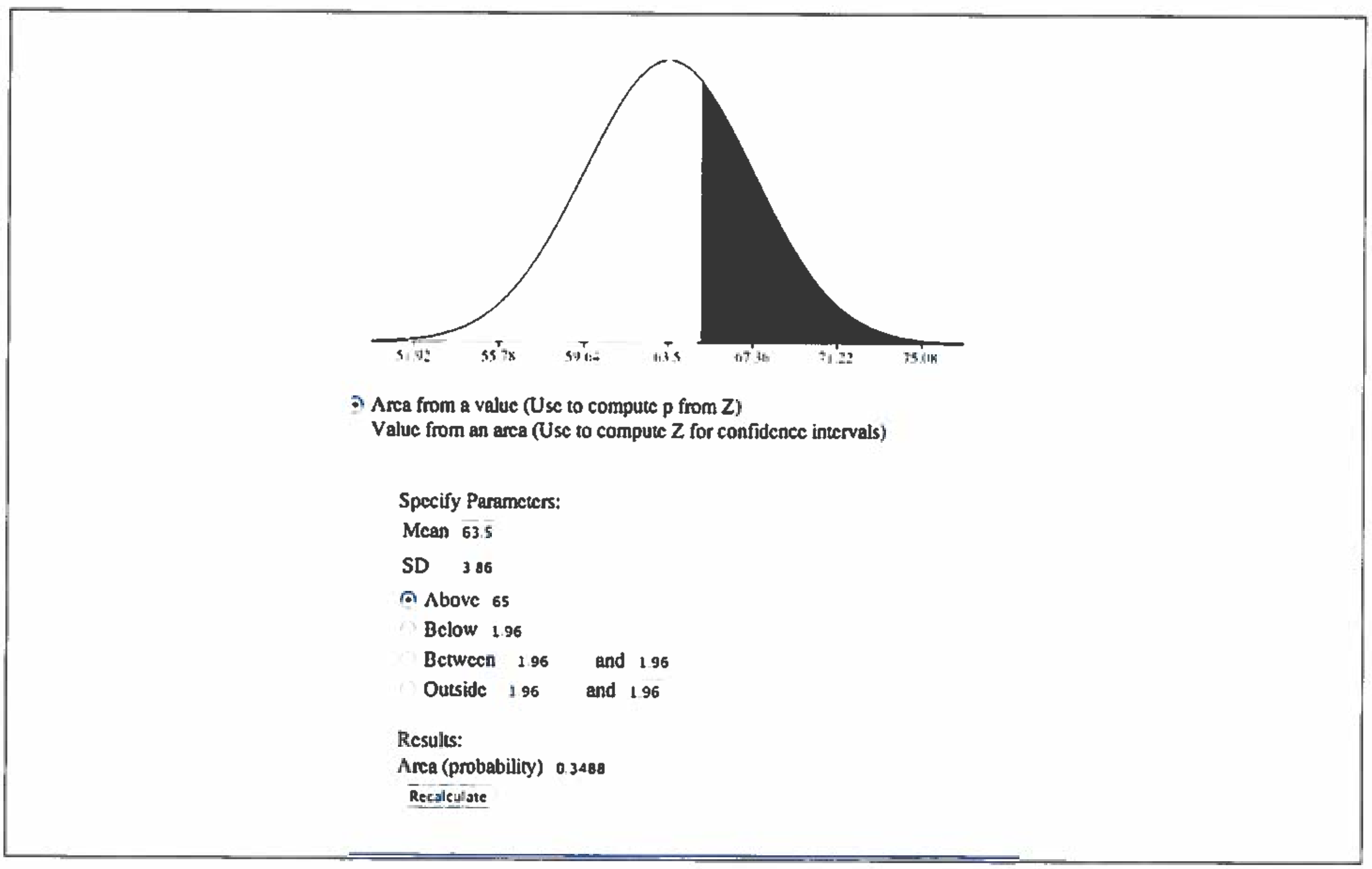
Click **on** the radio button to select, **"Area from a value (Use to compute p from Z)"**

Step 3

In the box, labeled **mean**, enter **63.5**, in the box labeled **SD** enter **3.86.**

Step 4

Click **on** the radio button to select, **"Above"** In the box at right, enter **65.** Click **recalculate**



Solution Using Z-Score:

$$\Pr\left[\,\bar{X}_{n=10} > 65\text{ g}\,\right] = \Pr\left[\,\text{Normal } \mu_{\bar{X}}=63.5 \;\; \sigma_{\bar{X}}=\frac{12.2}{\sqrt{10}} > 65\,\right]$$

$$= \Pr\left[\,\text{Standard normal} > \frac{65-\mu_{\bar{X}}}{\sigma_{\bar{X}}}\,\right] = \Pr\left[\,\text{Standard normal} > \frac{65-63.5}{12.2/\sqrt{10}}\,\right]$$

$$= \Pr\left[\,\text{Normal }(0,1) > 0.3888\,\right] = .3483$$